

An intrinsic characterization of semi-normal operators

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Abstract

Two necessary and sufficient conditions for an operator to be semi-normal are revealed. For a Volterra integration operator the set where the operator and its adjoint are metrically equal is described.

Let A be a linear bounded operator, acting in a Hilbert space $(\mathcal{H}, \langle \bullet, \bullet \rangle)$ and $W(A)$ denote the numerical range of A . If $C(A) = A^*A - AA^*$ is semi-definite, the operator A is said [Putnam \[1967\]](#) to be semi-normal, particularly, if $C(A) \geq \mathbf{0}$, then A is hyponormal. The well-known and important class of normal operators is characterized by the equality $AA^* = A^*A$. It is easy to see that the last condition is equivalent to the equality $\|Ax\| = \|A^*x\|$ for any $x \in \mathcal{H}$, meaning that any normal operator is metrically equal to its adjoint on all \mathcal{H} . For hyponormal operator in [Stampfli \[1966\]](#) is proved that conditions

$$\|Ax\| = \|A^*x\| \quad \text{and} \quad A^*Ax = AA^*x \quad (1)$$

are equivalent. Note that the set of points, satisfying the second condition is the null space of the self-commutator- $N(C(A))$. As the both conditions are symmetric, Stampfli's result remains valid for semi-normal operators. Using this property, Stampfli has shown that any extreme point of the numerical range of a hyponormal operator A is a reducing eigenvalue.

Denote

$$E(A) = \{x : \|Ax\| = \|A^*x\| \}$$

and

$$M_\lambda(A) = \{x : \langle Ax, x \rangle = \lambda \|x\|^2\}$$

Evidently conditions $\lambda \in W(A)$ and $M_\lambda(A) \neq \{\theta\}$ are equivalent.

Proposition 1. For any operator A one has

$$\|Ax\|^2 - \|A^*x\|^2 = \langle C(A)x, x \rangle,$$

particularly,

$$E(A) = M_0(C(A)).$$

Proof. As $\|Ax\|^2 = \langle Ax, Ax \rangle = \langle A^*Ax, x \rangle$ and $\|A^*x\|^2 = \langle AA^*x, x \rangle$, the conditions $\|Ax\| = \|A^*x\|$ and $\langle (C(A))x, x \rangle = 0$ are equivalent.

Proposition 2. The operator A is semi-normal if and only if 0 is an extreme point of the closure of $W(C(A))$.

Proof. As the numerical range of any self-adjoint operator is a convex subset of \mathbb{R} we have $\overline{W(C(A))} = [a; b]$. If 0 is an extreme point of $\overline{W(C(A))}$ then $ab = 0$, hence A is semi-normal. Let now A be semi-normal, i.e $ab \geq 0$. We show that the strict inequality $ab > 0$ is not possible. According to a result of Radjavi ([Radjavi \[1966\]](#), Corollary 1) if B is a selfadjoint operator such that $B \geq \alpha I$ ($B \leq -\alpha I$) for some positive number α , then B is not a self-commutator. Thus $ab = 0$ and 0 is an extreme point of $\overline{W(C(A))}$.

Proposition 3. The equivalence (1) is true if and only if the operator A is semi-normal.

Proof. Only the necessity of this condition should be proved. Let (1) be true. If $E(A) = \{\theta\}$, then $0 \notin W(C(A))$, hence it lies entirely in the positive or negative semi-axis. Let now x and y be two elements from $E(A)$. Then from $\|Ax\| = \|A^*x\|$, $\|Ay\| = \|A^*y\|$ follows $AA^*x = A^*Ax$, $AA^*y = A^*Ay$ and $AA^*(x+y) = A^*A(x+y)$, implying $\|A(x+y)\| = \|A^*(x+y)\|$. According to [Embry \[1970\]](#) the linearity of $M_\lambda(A)$ is equivalent to the condition that λ is an extreme point of $W(A)$. Thus 0 is an extreme point of $W(C(A))$, completing the proof.

Remark 1. The principal reason in the proof above was the linearity of $M_0(C(A))$. If the last condition is satisfied, then A is semi-normal and by Stampfli's result $E(A) = N(C(A))$.

Remark 2. In ([Gevorgyan \[2006\]](#), Proposition 2, Corollary 1) is proved that $N(A) = M_0(A)$ if and only if

$$A = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix},$$

$0 \notin W(B)$, where any direct summand may be absent.

The situation is more interesting for non semi-normal operators. The example below exhibits the matter for a non semi-normal quasinilpotent compact operator.

Example. Consider the Volterra integration operator V

$$(Vf)(x) = \int_0^x f(t)dt, f \in L^2(0; 1).$$

We have $V1 = x, V^*1 = 1 - x$, implying $\|V1\| = \|V^*1\|$. Let now $f \perp 1$. As $\int_0^x f(t)dt + \int_x^1 f(t)dt = \int_0^1 f(t)dt$, we have $Vf = -V^*f$ and $\|Vf\| = \|V^*f\|$, therefore $\{1, L^2(0; 1) \ominus 1\} \subset E(A)$.

The self-commutator of V is

$$(C(V)f)(x) = \int_0^1 f(t)dt - x \int_0^1 f(t)dt - \int_0^1 tf(t)dt$$

or

$$(C(V)f)(x) = \left(\frac{1}{2} - x\right) \int_0^1 f(t)dt + \int_0^1 \left(\frac{1}{2} - t\right) f(t)dt.$$

Denoting $e_1 = 1, e_2 = \sqrt{3}(1 - 2x)$ (they are two first $L^2(0; 1)$ -orthonormal polynomials) we get

$$C(V)f = \frac{1}{2\sqrt{3}} (\langle f, e_1 \rangle e_2 + \langle f, e_2 \rangle e_1).$$

Now we introduce two orthonormal elements

$$u_1 = \frac{1}{\sqrt{2}}(e_1 + e_2) = \sqrt{2 + \sqrt{3}} - \sqrt{6}x,$$

$$u_2 = \frac{1}{\sqrt{2}}(e_1 - e_2) = \sqrt{6}x - \sqrt{2 - \sqrt{3}},$$

and arrive at the canonical form of the self-commutator of V

$$C(V)f = \frac{1}{2\sqrt{3}} (\langle f, u_1 \rangle u_1 - \langle f, u_2 \rangle u_2). \quad (2)$$

Note that the product $u_1 u_2$ defines the third orthogonal polynomial $6x^2 - 6x + 1$.

From (2) follows that the spectrum of $C(V)$ is the set $\left\{-\frac{\sqrt{6}}{2}, 0, \frac{\sqrt{6}}{2}\right\}$, hence $W(C(V)) = \left[-\frac{\sqrt{6}}{2}; \frac{\sqrt{6}}{2}\right]$. The null-space of $C(V)$ consists of functions orthogonal to the first-order polynomials- $L^2(0; 1) \ominus \vee \{1, x\}$, where \vee denotes the linear span of the set. As

$$\langle C(V)f, f \rangle = \frac{1}{2\sqrt{3}} (|\langle f, u_1 \rangle|^2 - |\langle f, u_2 \rangle|^2),$$

we get $E(V) = \{f : |\langle f, u_1 \rangle| = |\langle f, u_2 \rangle|\}$, i.e.

$$E(V) = \bigcup_{\varphi \in [0; 2\pi)} L_\varphi$$

where L_φ is the orthocomplement to the subspace, generated by the element $u_1 - e^{i\varphi} u_2$.

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